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Escape to infinity under the action of a potential and a constant electromagnetic field

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Abstract
Escape to infinity is proved for a great variety of potentials, including the potential created by an infinite number of sources. Relativistic escape is studied. Escape in the presence of a constant electromagnetic field and a potential is also considered.

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1. Introduction

We deal in this paper with the problem of finding conditions guaranteeing that the Newtonian equations of motion of a unit-charge, unit-mass particle,

$$\left.\begin{array}{l} \ddot{\mathbf{x}} = -\nabla V(\mathbf{x}, t) + \varepsilon(\mathbf{x}) + \bar{\bar{A}}(\mathbf{x})\dot{\mathbf{x}} \\ \mathbf{x} \in R^n \quad (n = 2, 3) \qquad \mathbf{x} = (x_i) \\ \nabla = \text{gradient operator} \end{array}\right\} \tag{1}$$

admit *unbounded solutions* $\mathbf{x}(t; \mathbf{x}_0, \dot{\mathbf{x}}_0)$. That is the Euclidean norm of these solutions,

$$\|\mathbf{x}(t; \mathbf{x}_0, \dot{\mathbf{x}}_0)\| \tag{2}$$

is unbounded when $t > 0$ for initial conditions $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$ lying on open sets Δ of the phase space $\Im$ of equations (1). These unbounded solutions are called in theoretical physics and chemistry escape or scattering solutions [1].

Note that $\varepsilon(\mathbf{x})$ and $\bar{\bar{A}}(\mathbf{x})$ (see equation (1)) stand for a global vector field and a global antisymmetric (n, n) matrix taking into account the effect on the unit charge of a global steady electromagnetic field. Non-global steady electric fields are assumed to be described by the non-global potential $V(\mathbf{x})$ (think of the local electric field produced by a single charge or the Newtonian gravitational field originated in a single mass).

The time T ($T > 0$) such that $\|\mathbf{x}(t; \mathbf{x}_0, \dot{\mathbf{x}}_0)\|$ is bounded in $[0, T)$ but $\lim_{t\to T} \|\mathbf{x}(t; \mathbf{x}_0, \dot{\mathbf{x}}_0)\| = +\infty$, that is the time taken by the particle in reaching infinity, can be finite or infinite [2], but this point will *not* be treated here.

An escape problem in theoretical chemistry arises when an electron in an atom at $t_0 = 0$ goes to infinity for $t > 0$ and the original atom gets 'ionized'.

When $V(\mathbf{x}) = V(\|\mathbf{x}\|)$, V is called a central potential. If, moreover, $\varepsilon = \mathbf{0}$, $\bar{\bar{A}} = 0$ the problem of finding an open set Δ, $\Delta \subset \Im$, such that $(\mathbf{x}_0, \dot{\mathbf{x}}_0) \in \Delta$ implies

$$\|\mathbf{x}(t; \mathbf{x}_0, \dot{\mathbf{x}}_0)\| = \text{unbounded} \qquad \text{for} \quad t = T \tag{3}$$

is trivial, since equation (1) is in this case integrable [3]. When $V(\mathbf{x})$ is not central equation (1) is, in general, not integrable ($\varepsilon = \mathbf{0}$, $\bar{\bar{A}} = 0$) and escape to infinity must be decided in other ways. In fact, the problem of existence of scattering solutions of equation

$$\ddot{\mathbf{x}} = -\nabla V(\mathbf{x}) \tag{4}$$

was studied [4] in R^3 when $V(\mathbf{x})$ can be expressed as a linear combination of Newton-like potentials ($V \sim \frac{1}{\rho}$, $\rho = \|\mathbf{x}\|$, when $\rho \to +\infty$). The basic idea in [4] was to associate with equation (4) a differential inequality for ρ, $\dot{\rho}$ and $\ddot{\rho}$.

Using basically the same idea of differential inequalities we get in this paper the following results:

(i) Conditions guaranteeing escape to infinity when $V(\mathbf{x}, t)$ is a finite superposition of Newtonian [4], Lennard-Jones [5], Morse [6], Yukawa [7], Saxon–Woods [8], Eckart [9] and Hylleraas [10] potentials created by point sources at the positions $\mathbf{x}_i(t)$ ($i = 1, \ldots, N$) satisfying

$$\|\mathbf{x}_i(t)\| \leqslant M \qquad M \in R^+ \quad \text{(positive real number)} \tag{5}$$

are obtained in section 2.

(ii) Perturbation of central potentials and escape are studied in section 3.

(iii) Relativistic escape to infinity for the cases studied in sections 2 and 3 is considered in section 4.

(iv) Escape under the presence of a constant electromagnetic field (ε, $\mathbf{B}$) and a central potential has been considered in section 5.

The problem of escape to infinity remains open for the relativistic equations

$$\frac{\mathrm{d}(\gamma\dot{\mathbf{x}})}{\mathrm{d}t} = -\nabla V(\mathbf{x}, t) + \varepsilon + \bar{\bar{A}}\dot{\mathbf{x}} \qquad \gamma = (1 - \dot{\mathbf{x}}^2)^{-1/2} \quad \text{light speed} = 1 \tag{6}$$

for which the methods of sections 2–5 seem to fail (see section 6).

The conclusions and open problems on escape to infinity appear in section 6. The proof of the main theorem used in this paper to ascertain escape to infinity is given in the appendix. Escape in the presence of a potential has been studied in [11] when $V(\mathbf{x})$ is non-Newtonian. A careful mathematical study of Newtonian potentials can be found in [12].

2. Escape to infinity when $V(\mathbf{x}, t)$ is a linear superposition of central potentials and $\varepsilon = \mathbf{0}$, $\bar{\bar{A}} = 0$

We now consider the case of a potential $V(\mathbf{x}, t)$ of the form

$$V(\mathbf{x}, t) \stackrel{\text{def}}{=} \sum_{i=1}^{N} V_i(\|\mathbf{x} - \mathbf{x}_i(t)\|) \tag{7}$$

where V_i stands for a central potential originated in the source $\mathbf{x}_i(t)$ ($i = 1, \ldots, N$); think of the solar system, where $\mathbf{x}_i(t)$ are the instantaneous positions of the Sun and the larger planets

($N = 10$) or of an electron subjected to the action of several positive charges. We can also assume that V_i is analytic on its domain of definition.

Applications of the results in this section can be found in the comments between formulae (16) and (17). Throughout this section we assume that equation (5) holds, that is $\|\mathbf{x}_i(t)\|$ is assumed to be bounded when $t > 0, \forall i = 1, \ldots, N$. In addition to this the functions V_i are assumed to satisfy

$$V_i(u) < K_i \in R \text{ for large } u. \tag{8a}$$

$$\frac{\mathrm{d}V_i(u)}{\mathrm{d}u} \equiv V_i'(u) \text{ is positive or negative for large } u \text{ (that is, the sign of } V_i'(u) \text{ is constant at infinity)} \tag{8b}$$

$$\frac{\mathrm{d}^2 V_i(u)}{\mathrm{d}u^2} \equiv V_i''(u) \text{ is negative for large } u. \tag{8c}$$

where u stands for $\|\mathbf{x} - \mathbf{x}_i\|$.

Under these conditions (that is when equations (5) and (8) are satisfied), we shall show that the equation

$$\ddot{\mathbf{x}} = -\nabla\left(\sum_{i=1}^{N} V_i(\|\mathbf{x} - \mathbf{x}_i(t)\|)\right) \tag{9}$$

admits escape or scattering solutions (in open sets of phase space).

At the end of this section escape to infinity, when an infinite number of pointlike sources are present, is briefly studied.

2.1. Escape to infinity: a finite number of central potentials

Consider the differential equation (9). Taking two derivatives with respect to t on both sides of the equality $\rho^2 = \mathbf{x} \cdot \mathbf{x}$ ($\rho = \sqrt{\mathbf{x} \cdot \mathbf{x}}$) and using equation (9) we easily get

$$\ddot{\rho} \geqslant \frac{-\nabla V(\mathbf{x}, t) \cdot \mathbf{x}}{\rho} \tag{10}$$

and since V is a linear superposition of central potentials V_i equation (10) becomes

$$\ddot{\rho} \geqslant \left(-\sum_{i=1}^{N} \frac{\mathbf{x} - \mathbf{x}_i(t)}{\|\mathbf{x} - \mathbf{x}_i(t)\|} \cdot V_i'\right) \cdot \frac{\mathbf{x}}{\rho} = \frac{-1}{\rho}\left(\sum_{i=1}^{N} \frac{\rho^2 - \mathbf{x} \cdot \mathbf{x}_i(t)}{\|\mathbf{x} - \mathbf{x}_i(t)\|} \cdot V_i'\right) \tag{11}$$

We just consider now the case $V_i' \geqslant 0$ ($\forall i$) when ρ is large (see equation (8b)), since the other cases can be reduced to this one.

Under these conditions equation (11) becomes

$$\ddot{\rho} \geqslant -\sum_{i=1}^{N} V_i'(\|\mathbf{x} - \mathbf{x}_i(t)\|) \qquad V_i' > 0 \quad \forall i \quad \text{for large } \rho. \tag{12}$$

Now taking into account equation (5) we can write (when ρ is large)

$$\|\mathbf{x} - \mathbf{x}_i(t)\| \geqslant \|\mathbf{x}\| - \|\mathbf{x}_i(t)\| \geqslant \rho - M \tag{13}$$

and since by equation (8c) V_i' decreases for large ρ we can write

$$V_i'(\|\mathbf{x} - \mathbf{x}_i(t)\|) \leqslant V_i'(\rho - M) \tag{14}$$

Table 1. Classical potentials.

	$V(u)$	K	$V'(u)$	Sign of $V'(u)$ for large u	$V''(u)$	Sign of $V''(u)$ for large u
Newton	$-\frac{1}{u}$	0	$\frac{1}{u^2}$	+	$\frac{-2}{u^3}$	−
Morse	$-\mathrm{e}^{-(u-1)^2}$	0	$2(u-1)\,\mathrm{e}^{-(u-1)^2}$	+	$-2\,\mathrm{e}^{-(u-1)^2}[2(u-1)^2-1]$	−
Lennard-Jones	$\frac{1}{u^{12}}-\frac{1}{u^6}$	0	$\frac{-12}{u^{13}}+\frac{6}{u^7}$	+	$\frac{-6}{u^8}\left(7-\frac{26}{u^6}\right)$	−
Saxon–Woods	$\frac{-1}{1+\mathrm{e}^{u-1}}$	0	$\frac{\mathrm{e}^{u-1}}{(1+\mathrm{e}^{u-1})^2}$	+	$\frac{\mathrm{e}^{u-1}(1-\mathrm{e}^{u-1})}{(1+\mathrm{e}^{u-1})^3}$	−
Yukawa	$-\frac{\mathrm{e}^{-u}}{u}$	0	$\frac{(1+u)\mathrm{e}^{-u}}{u^2}$	+	$-\frac{\mathrm{e}^{-u}(u^2+2u+2)}{u^3}$	−
Eckart	$\frac{-\mathrm{e}^{-u}}{1+\mathrm{e}^{-u}}$	0	$\frac{\mathrm{e}^{-u}}{(1+\mathrm{e}^{-u})^2}$	+	$\frac{-\mathrm{e}^{-u}(1-\mathrm{e}^{-u})}{(1+\mathrm{e}^{-u})^3}$	−
Hylleraas	$\frac{-1}{\cosh^2 u}$	0	$\frac{2\sinh u}{\cosh^3 u}$	+	$\frac{2(1-\sinh^2 u)}{\cosh^4 u}$	−

Note that the positive coupling constant k_i (the 'mass' of the attracting centre when the potential is Newtonian) has been omitted in the column of $V(u)$.

and equation (12) becomes

$$\ddot{\rho} \geqslant -\sum_{i=1}^{N} V_i'(\rho - M) \stackrel{\text{def}}{=} -\frac{\mathrm{d}W(\rho)}{\mathrm{d}\rho} \tag{15}$$

where $W(\rho)$ stands for

$$W(\rho) = \int \left(\sum_{i=1}^{N} V_i'(\rho - M)\right) \mathrm{d}\rho = \sum_{i=1}^{N} V_i(\rho - M). \tag{16}$$

Therefore when (8*a*) holds $W(\rho)$ (see equation (16)) is bounded. We shall immediately see the consequences of this on the solutions of (15).

The reader can see in table 1 how the hypotheses assumed up to now are fulfilled by certain classical potentials.

We can see in table 1 that the Newton, Morse, Lennard-Jones, Saxon–Woods, Yukawa, Eckart and Hylleraas potentials satisfy conditions (8*a*)–(8*c*) and therefore escape to infinity in the presence of one, or several, of these potentials is guaranteed.

Potentials such as $\mathrm{e}^{-u}\sin u$ or $\frac{\sin u}{1+u^2}$ violate condition (8*b*). Escape to infinity under the action of these potentials must be studied by other methods.

It is well known that the Newtonian potential $\frac{-1}{u}$ rules the classical motion of celestial bodies [4]; it also rules the motion of classical charged particles under the influence of charges of opposite sign (via a Coulomb potential). The Morse potential [6] accounts for the interaction between nuclei in diatomic molecules, while the Lennard-Jones potential [5] was proposed as a model for the interaction between noble gas atoms. The Yukawa [7] and Saxon-Woods potentials [8] arise in order to model the low energy nuclear forces between protons and neutrons in nuclei. Finally, the Eckart potential [9] is useful in the description of the motion of electrons in a potential barrier.

Note that the results of this section hold when the potentials V_i of equation (7) are different or equal for different values of i.

Consider now (see equation (15)) the differential inequality

$$\ddot{\rho} \geqslant -\frac{\mathrm{d}W(\rho)}{\mathrm{d}\rho} \qquad W(\rho) = \text{bounded} \tag{17}$$

and the differential equation

$$\ddot{\rho} = -\frac{\mathrm{d}W(\rho)}{\mathrm{d}\rho}. \tag{18}$$

Now, equation (18) admits unbounded solutions (remember that $W(\rho) \leqslant K, K \in R$ and $\frac{1}{2}\dot{\rho}^2 + W(\rho)$ is a first integral of equation (18)).

On the other hand, when $V_i'' < 0$ $(\forall i)$ when ρ is large (assumption (8c)) we get the following result (see the appendix): if $(\rho_0, \dot{\rho}_0)$ are initial conditions for equation (18) and

(i) the corresponding solution $\rho(t)$ is unbounded for $t > 0$,
(ii) $W''(\rho)$ is negative for large ρ,

then the corresponding solution $\tilde{\rho}(t)$ of equation

$$\ddot{\rho} = a(\rho) + p(t) \qquad p(t) \geqslant 0 \qquad p(0) > 0 \tag{19}$$

(where $a(\rho)$ stands for $-W'$), with the same initial conditions $(\rho_0, \dot{\rho}_0)$ as equation (18), is also unbounded for $t > 0$.

It follows that since equation (18) possesses unbounded solutions for initial conditions $(\rho_0, \dot{\rho}_0)$ lying on open sets, the same result holds for equation (17) (see the appendix for a more detailed explanation).

2.2. *Escape to infinity: an infinite number of central potentials*

To conclude this section we briefly study escape to infinity when an *infinite* system of central potentials is present. That is, we assume that $V(\mathbf{x}, t)$ is given by

$$V(\mathbf{x}, t) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} V_i(\|\mathbf{x} - \mathbf{x}_i(t)\|). \tag{20}$$

The force associated with $V(\mathbf{x}, t)$ is

$$\mathbf{F}(\mathbf{x}) = -\sum_{i=1}^{\infty} \left(V_i'(\|\mathbf{x} - \mathbf{x}_i(t)\|) \frac{\mathbf{x} - \mathbf{x}_i(t)}{\|\mathbf{x} - \mathbf{x}_i(t)\|} \right). \tag{21}$$

Uniform convergence of series (20) and (21) for large ρ is, of course, assumed. On the other hand $V(\mathbf{x}, t)$ is also assumed to be bounded for large ρ:

$$V(\mathbf{x}, t) \leqslant K \qquad \text{for large } \rho. \tag{22}$$

All these hypotheses can be discussed when the V_i are given. For instance, when $V_i(u)$ is the Newtonian potential,

$$V_i(u) = -\frac{m_i}{u} \qquad (m_i > 0) \tag{23}$$

it can immediately be proved that the assumptions

$$\|\mathbf{x}_i\| \stackrel{i\to\infty}{\to} +\infty \qquad \sum_{i=1}^{\infty} m_i \|\mathbf{x}_i\|^{-1} = \text{finite} \qquad \sum_{i=1}^{\infty} m_i \|\mathbf{x}_i\|^{-2} = \text{finite} \tag{24}$$

imply the uniform (and absolute) convergence of the series appearing in equations (20) and (21) (see [4]). Note that the first of equations (24) guarantees that the masses m_i do *not accumulate* around any $\mathbf{x} \neq \mathbf{x}_i$.

Concerning escape to infinity when there is an infinite number of attracting centres (see equations (20) and (21)), we shall limit ourselves to show that escape *is possible* for some particular cases. The discussion of the general case (see equations (20) and (21)) is extremely tedious and will not be given.

Case 1. One of these particular cases is that in which there exists a plane (say $z = 0$) such that $z_i(t) < 0$ ($\forall t > 0$). Then escape to infinity is possible (in fact one can escape to infinity by the region $z > 0$).

Indeed, we get from equation (21)

$$\ddot{z} = -\sum_{i=1}^{\infty} V_i'(\|\mathbf{x} - \mathbf{x}_i(t)\|)\frac{z - z_i(t)}{\|\mathbf{x} - \mathbf{x}_i(t)\|} \tag{25}$$

and since $z > 0$, $z_i(t) < 0$ and $V_i'(u) > 0$ at infinity (see the restrictions appearing in (8*a*)–(8*c*)), we can write

$$\ddot{z} \geqslant -\sum_{i=1}^{\infty} V_i'(\|\mathbf{x} - \mathbf{x}_i(t)\|) \tag{26}$$

and on account of equation (14) we get

$$\ddot{z} \geqslant -\sum_{i=1}^{\infty} V_i'(\rho - M) \overset{z\text{-axis}}{=} -\sum_{i=1}^{\infty} V_i'(z - M) \overset{\text{def}}{=} -\frac{\mathrm{d}W(z)}{\mathrm{d}z} \tag{27}$$

where

$$W(z) \overset{\text{def}}{=} \int \sum_{i=1}^{\infty} V_i'(z - M)\,\mathrm{d}z = \sum_{i=1}^{\infty} V_i(z - M) \tag{28}$$

(for the mathematical justification concerning the commutation of integrals and series, see [13]) and since equation (8*a*) holds, $W(z)$ is bounded since $V(\mathbf{x}, t)$ was assumed to be bounded at infinity (see equation (22)). Therefore escape to infinity is guaranteed.

Case 2. We now assume that there is an axis (say the z-axis) such that for every i there exists i^* such that $\mathbf{x}_{i^*}(t) = (-x_i(t), -y_i(t), z_i(t))$ and $V_i(\|\mathbf{x} - \mathbf{x}_i\|) = V_{i^*}(\|\mathbf{x} - \mathbf{x}_{i^*}\|)$. When these conditions hold the z-axis is an invariant line. Escape to infinity is also possible in this case (in fact via the z-axis). The proof is similar to that of case 1 and will not be given here.

3. Escape and perturbations

Consider that the unit-mass, unit-charge particle is subject to the action of a potential of the form

$$V(\rho) + V_P(\mathbf{x}) \qquad \frac{\|\nabla V_P(\mathbf{x})\|}{|V'(\rho)|} \leqslant C \in R \qquad \text{for large } \rho \tag{29}$$

where V_P stands for the perturbation of $V(\rho)$, $V(\rho)$ being a central potential, and $V'(\rho) = \frac{\mathrm{d}V(\rho)}{\mathrm{d}\rho}$. As usual V and V_P are assumed to be analytic on their definition domains. We also assume that V is bounded and that $V'(\rho) > 0$, $V''(\rho) < 0$ for large values of ρ (see equations (8)).

Under these hypotheses let us see that we get escape to infinity in equation

$$\ddot{\mathbf{x}} = -\nabla(V(\rho) + V_P(\mathbf{x})). \tag{30}$$

One application of the results of this section is the study of the effects on escape of the high-order residual multipole perturbations of a Coulombian or Newtonian non-pointlike charge, or mass, distribution. Note that escape occurs under the presence of just the main (isotropic)

Table 2. Central potentials V and perturbations V_P.

V	V_P	Bound of V for large ρ	$\frac{\|\nabla V_P\|}{\|V'\|}$
$-\frac{1}{\rho}$	$\sum_{i=1}^{m} \frac{B_i(\rho,\theta)}{\rho^{i+1}}$, B_i, $B_{i,\rho}$, $B_{i,\theta}$ bounded and periodic in θ	0	$\approx\|B_{1,\rho}\| \to$ bounded when ρ is large
$-\frac{1}{\rho}$	$\rho\cos\theta$, constant electric field	0	$\rho^2 \to$ not bounded when ρ is large

term in the potential $(-1/\rho)$, when, as is well known, the total energy E of the test unit-charge, unit-mass particle is positive.

Indeed, proceeding as in section 2 we get for large ρ

$$\ddot{\rho} \geqslant -\frac{\mathrm{d}}{\mathrm{d}\rho}((1+C)V(\rho)) \tag{31}$$

and escape is guaranteed (see the appendix) since $(1+C)V(\rho)$ is bounded when ρ is large (see the lines following equation (29)).

The reader can have a look at the list (table 2) of central potentials V and perturbations V_P. Note that ρ and θ are the standard polar coordinates in R^2. The example in the second row of table 2 does not satisfy condition (29), and is separately studied in section 5. Note that the function V_P in the first example of table 2 includes (as particular cases) the dipole potential $(i = 1, B_1 = \cos\theta)$ and the multipole potentials when $B_i = B_i(\theta)$ [14]. Another case of escape to infinity under perturbations is given now. Under perturbation of the potential $V(\mathbf{x}, t)$ of equation (7) the total potential is $V(\mathbf{x}, t) + V_P(\mathbf{x})$.

Defining V_p small when

$$\|\nabla V_P\| \leqslant C\|\nabla V\| \qquad C \in R \quad \rho = \text{large} \tag{32}$$

we immediately get the following inequality,

$$\ddot{\rho} \geqslant -(1+C)\|\nabla V(\mathbf{x}, t)\| \tag{33}$$

and writing V explicitly we get (see equations (7) and (8) at the beginning of section 2 and equations (15) and (16))

$$\ddot{\rho} \geqslant -(1+C)\left(\sum_{i=1}^{N} V_i'(\|\mathbf{x} - \mathbf{x}_i(t)\|)\right) \geqslant -(1+C)\left(\sum_{i=1}^{N} V_i'(\rho - M)\right) \stackrel{\text{def}}{=} -\frac{\mathrm{d}W(\rho)}{\mathrm{d}\rho}. \tag{34}$$

Acting now with $W(\rho)$ as in section 2 we get after straightforward manipulations that escape to infinity is possible when equation (32) holds.

A simple example for which equation (32) holds arises when V_P is a multipole potential (going to zero at infinity as $\frac{1}{\rho^n}$, $n \geqslant 2$) and $V(\rho)$ is the Newtonian potential $-\frac{1}{\rho}$. In this case $\|\nabla V_P\|$ behaves (when ρ is large) like $\rho^{-(1+n)}$ and $\|\nabla V\|$ like ρ^{-2} and it is clear that equation (32) holds.

4. Relativistic escape

We prove in this section that relativistic escape to infinity occurs for all the cases studied in sections 2 and 3. Note that throughout this section we assume $\varepsilon(\mathbf{x}) = \mathbf{0}$, $\bar{\bar{A}}(\mathbf{x}) = 0$.

We study in section 4.1 the case of a potential $V(\mathbf{x}, t)$ which is a finite superposition of central potentials. An infinite number of central potentials are studied in sections 4.2 and 4.3.

Relativistic perturbations are considered in section 4.4. Finally, in section 4.5 we show that escape to infinity is possible under the action of the magnetic field created by a monopole.

The main interest of this section is to show that high energy escape to infinity can be handled as well via the methods of the preceding sections 2 and 3.

The relativistic equation of motion is now

$$\left.\begin{array}{l} \dfrac{\mathrm{d}}{\mathrm{d}t}(\gamma\dot{\mathbf{x}}) = -\nabla V(\mathbf{x}, t) \\ \gamma = (1-\dot{\mathbf{x}}^2)^{-1/2} \\ c = \text{ light speed } = 1 \\ \mathbf{x} = (x, y, z) \in R^3 \end{array}\right\} . \tag{35}$$

Writing the second-order differential system (35) in normal form, we get

$$\ddot{\mathbf{x}} = (1-\dot{\mathbf{x}}^2)^{1/2} \cdot \bar{\bar{M}} \cdot (-\nabla V(\mathbf{x}, t)) \tag{36}$$

$\bar{\bar{M}}$ being the matrix defined by

$$\bar{\bar{M}} = \begin{pmatrix} 1-\dot{x}^2 & -\dot{x}\dot{y} & -\dot{x}\dot{z} \\ -\dot{x}\dot{y} & 1-\dot{y}^2 & -\dot{y}\dot{z} \\ -\dot{x}\dot{z} & -\dot{y}\dot{z} & 1-\dot{z}^2 \end{pmatrix}. \tag{37}$$

Note that the entries m_{ij} of $\bar{\bar{M}}$ satisfy $|m_{ij}| \leqslant 1$.

4.1. Relativistic escape: a finite number of central potentials

We assume in this subsection that restriction (5) holds and that equation (36) becomes

$$\ddot{\mathbf{x}} = (1-\dot{\mathbf{x}}^2)^{1/2}\bar{\bar{M}} \cdot (-\nabla V(\mathbf{x}, t)) \qquad -\nabla V(\mathbf{x}, t) = -\sum_{i=1}^{N} \frac{\mathbf{x}-\mathbf{x}_i(t)}{\|\mathbf{x}-\mathbf{x}_i(t)\|} \cdot V_i'(\|\mathbf{x}-\mathbf{x}_i(t)\|). \tag{38}$$

We show that escape to infinity occurs. The reader will note that this is proved without invoking energy conservation.

Taking into account that $|m_{ij}| \leqslant 1$ we get

$$\mathbf{x}\bar{\bar{M}}\mathbf{F}(\mathbf{x}) \leqslant 3\|\mathbf{x}\| \cdot \|\mathbf{F}(\mathbf{x})\| = 3\rho\|\mathbf{F}(\mathbf{x})\| \tag{39}$$

and proceeding as in section 2 we get (recall that $V_i' > 0$ for large ρ)

$$\ddot{\rho} \geqslant -3\sum_{i=1}^{N} V_i'(\|\mathbf{x}-\mathbf{x}_i(t)\|) \tag{40}$$

and since $V_i'' < 0$ for large ρ (see equation (8)) equation (40) becomes

$$\ddot{\rho} \geqslant -3\sum_{i=1}^{N} V_i'(\rho - M) \stackrel{\text{def}}{=} -\frac{\mathrm{d}W}{\mathrm{d}\rho} \tag{41}$$

where

$$W = \int \left(\sum_{i=1}^{N} V_i'(\rho - M)\right) \mathrm{d}\rho = \sum_{i=1}^{N} V_i(\rho - M). \tag{42}$$

At this point it is easy to see that when (8) holds then $W(\rho)$ is bounded at infinity (see section 2), and therefore escape to infinity occurs.

4.2. Relativistic escape: an infinite number of central potentials

We now study the relativistic analogue of case 1 in section 2.2, that is, the potential $V(\mathbf{x}, t)$ is now created by an *infinite* number of pointlike sources in motion below the $z = 0$ plane. See equations (24) concerning the restrictions that the single potentials $V_i(\|\mathbf{x} - \mathbf{x}_i(t)\|)$ must satisfy in order that the series $\sum_{i=1}^{\infty} V_i(\|\mathbf{x} - \mathbf{x}_i(t)\|) \stackrel{\text{def}}{=} V(\mathbf{x}, t)$ be convergent.

Instead of equation (25) we must now write

$$\ddot{z} = -(1 - \dot{\mathbf{x}}^2)^{1/2} \sum_{i=1}^{\infty} (-\dot{x}\dot{z}, -\dot{y}\dot{z}, 1 - \dot{z}^2) \cdot \frac{(x - x_i(t), y - y_i(t), z - z_i(t))}{\|\mathbf{x} - \mathbf{x}_i(t)\|} \cdot V_i'(\|\mathbf{x} - \mathbf{x}_i(t)\|) \tag{43}$$

and taking into account that $V_i' > 0$ and $V_i'' < 0$ for large ρ, and $z_i(t) < 0\,\forall i$, after easy manipulations we get

$$\ddot{z} \geqslant -\sqrt{3} \sum_{i=1}^{\infty} V_i'(z - M) = -\frac{\mathrm{d}W(z)}{\mathrm{d}z}. \tag{44}$$

Therefore $W(z)$ is given by

$$W(z) = \sqrt{3} \int \left(\sum_{i=1}^{\infty} V_i'(z - M) \right) \mathrm{d}z = \sqrt{3} \sum_{i=1}^{\infty} V_i(z - M) \tag{45}$$

(for the mathematical justification concerning the commutation of integrals and series, see [13]). Making explicit the coupling constants k_i underlying the potentials of Newtonian type V_i,

$$V_i(u) = k_i V(u) \tag{46}$$

we get

$$W(z) = \sqrt{3} \left(\sum_{i=1}^{\infty} k_i \right) V(z - M). \tag{47}$$

Therefore when $\sum_{i=1}^{\infty} k_i$ is finite and $V(z)$ is bounded, $W(z)$ will also be bounded and escape to infinity is assured. Moreover, we can say that the z coordinate will increase without bound.

4.3. Relativistic escape: an infinite number of central potentials (continued)

Consider now case 2 studied at the end of section 2.2 of an *infinite* system of sources $(k_i, \mathbf{x}_i(t))$ symmetrical with respect to the z-axis.

In this case the z-axis is an invariant set and we can therefore write (see equation (43) for $x = y = 0, \dot{x} = \dot{y} = 0$)

$$\begin{aligned} \ddot{z} &\geqslant -(1 - \dot{z}^2)^{3/2} \sum_{i=1}^{\infty} \frac{z - z_i(t)}{\|\mathbf{x} - \mathbf{x}_i(t)\|} \cdot V_i'(\|\mathbf{x} - \mathbf{x}_i(t)\|) \\ &\geqslant -\sum_{i=1}^{\infty} V_i'(z - z_i) \geqslant -\sum_{i=1}^{\infty} V_i'(z - M) \stackrel{\text{def}}{=} -\frac{\mathrm{d}W(z)}{\mathrm{d}z} \end{aligned} \tag{48}$$

where

$$W(z) = \int \left(\sum_{i=1}^{\infty} V_i'(z - M) \right) \mathrm{d}z = \sum_{i=1}^{\infty} V_i(z - M). \tag{49}$$

Escape to infinity is ruled by the behaviour of $W(z)$ when z is large. For example, when $V_i(u) = k_i V(u)$ (see equation (46)) equation (49) becomes

$$W(z) = \left(\sum_{i=1}^{\infty} k_i\right) V(z - M). \tag{50}$$

Therefore $W(z)$ will be bounded at infinity whenever $\sum_{i=1}^{\infty} k_i$ is finite and $V(z)$ is bounded for large values of z. Note that by equation (8) this requirement on $V(z)$ is automatically satisfied.

4.4. Relativistic escape: perturbations

Consider now the case of the relativistic equation

$$\ddot{\mathbf{x}} = (1 - \dot{\mathbf{x}}^2)^{1/2} \bar{\bar{M}} \cdot \mathbf{F}(\mathbf{x}) \qquad \mathbf{F} = -\nabla(V(\rho) + V_P(\mathbf{x})) \tag{51}$$

where $V(\rho)$ is bounded and $V'(\rho) > 0$, $V''(\rho) < 0$ when ρ is large and V_P (the perturbation) satisfies

$$\|\nabla V_P\| \leqslant C|V'(\rho)| \qquad \text{for large } \rho. \tag{52}$$

Let us show that equation (51) admits escape solutions.

Indeed, from equation (51) we get

$$\left.\ddot{\rho} \geqslant -3(1 + C)V'(\rho) \stackrel{\text{def}}{=} -\frac{\mathrm{d}W(\rho)}{\mathrm{d}\rho}\right\} \tag{53}$$

where

$$W(\rho) = 3(1 + C)V(\rho) \tag{54}$$

which is bounded since $V(\rho)$ has been assumed to be bounded when ρ is large. Therefore equation (51) admits escape solutions.

We now study the case of the main potential $V(\mathbf{x}, t)$ being created by a *finite* collection of point sources at $\mathbf{x}_i(t)$, $\|\mathbf{x}_i(t)\| \leqslant M$ (see equation (7)) under perturbation of the potential $V_P(\mathbf{x})$.

Assuming that the perturbation V_P of V satisfies an equation of type (52), that is

$$\|\nabla V_P\| \leqslant C\|\nabla V\| \qquad C \in R \quad \rho = \text{large} \tag{55}$$

we get

$$\ddot{\rho} \geqslant -3(1 + C)\|\nabla V(\mathbf{x}, t)\| \tag{56}$$

where the expression for $\nabla V(\mathbf{x}, t)$ appears in equation (38). Therefore (remember that $V_i' > 0$ for large ρ) we can write

$$\|\nabla V(\mathbf{x}, t)\| \leqslant \sum_{i=1}^{N} V_i'(\|\mathbf{x} - \mathbf{x}_i(t)\|) \tag{57}$$

and equation (56) becomes (taking into account that $\|\mathbf{x}_i(t)\| \leqslant M$ and $V_i'' < 0$ when ρ is large)

$$\ddot{\rho} \geqslant -3(1 + C)\sum_{i=1}^{N} V_i'(\rho - M) \stackrel{\text{def}}{=} -\frac{\mathrm{d}W(\rho)}{\mathrm{d}\rho} \tag{58}$$

where

$$W(\rho) = 3(1 + C)\sum_{i=1}^{N} V_i(\rho - M). \tag{59}$$

Therefore, as $V_i(u)$ are assumed to be bounded, $W(\rho)$ is also bounded and escape to infinity is guaranteed.

4.5. Relativistic escape under the action of monopoles

Let us finally show that the high energy motion of an electric charge in the magnetic field *created* by a monopole [15] admits escape to infinity.

Indeed, the equation of motion is

$$\ddot{\mathbf{x}} = (1 - \dot{\mathbf{x}}^2)^{1/2} \bar{\bar{M}} \frac{\mathbf{x} \wedge \dot{\mathbf{x}}}{\|\mathbf{x}\|^3} \qquad \mathbf{x} \in R^3 \tag{60}$$

and therefore equation (10) becomes

$$\ddot{\rho} \geqslant \frac{\mathbf{x}}{\rho} \left((1 - \dot{\mathbf{x}}^2)^{1/2} \bar{\bar{M}} \frac{\mathbf{x} \wedge \dot{\mathbf{x}}}{\|\mathbf{x}\|^3} \right) \qquad \rho = \|\mathbf{x}\| \tag{61}$$

and, as it is not difficult to show that $\mathbf{x}\bar{\bar{M}}(\mathbf{x} \wedge \dot{\mathbf{x}})$ vanishes, we get

$$\ddot{\rho} \geqslant 0 \tag{62}$$

and escape to infinity is assured.

5. Escape under the action of a constant and uniform electric field and a central potential

Let $\mathbf{x} \in R^3$, $\varepsilon = (0, 0, \varepsilon)$ a constant electric field along the z-axis and $V(\rho)$ a central potential, $\rho = \|\mathbf{x}\|$ (classical Stark effect [16]). Sufficient conditions on $V(\rho)$ for escape to infinity are now given. The Newtonian differential equation is (in spherical coordinates)

$$\ddot{\mathbf{x}} = -\nabla V(\rho) + \varepsilon = -\nabla(V(\rho) + \rho\cos\theta) \qquad \varepsilon = (0, 0, \varepsilon). \tag{63}$$

Since any plane through the z-axis is invariant under (63) we just consider the differential equation

$$\ddot{\widehat{\boldsymbol{\rho}}} = -\nabla V(\widehat{\rho}) + \varepsilon \qquad \varepsilon = (\varepsilon, 0) \quad \widehat{\rho} = (x, y) \quad \widehat{\rho} = \sqrt{x^2 + y^2} \tag{64}$$

induced by (63) on any of these invariant planes.

Acting as in section 2 (equation (10)) we get from (64) (in the standard polar coordinates $(\widehat{\rho}, \widehat{\theta})$ of R^2)

$$\ddot{\widehat{\rho}} \geqslant -V'(\widehat{\rho}) + \varepsilon\cos\widehat{\theta} \tag{65}$$

and since the energy E is conserved we get

$$V(\widehat{\rho}) - \varepsilon\widehat{\rho}\cos\widehat{\theta} \leqslant E. \tag{66}$$

By eliminating $\cos\widehat{\theta}$, equations (65) and (66) imply

$$\ddot{\widehat{\rho}} \geqslant -V'(\widehat{\rho}) + \frac{V(\widehat{\rho}) - E}{\widehat{\rho}} \stackrel{\text{def}}{=} -\frac{\mathrm{d}W(\widehat{\rho})}{\mathrm{d}\widehat{\rho}} \tag{67}$$

$W(\widehat{\rho})$ being defined by

$$W(\widehat{\rho}) = V(\widehat{\rho}) - \int \frac{V(\widehat{\rho})}{\widehat{\rho}} \,\mathrm{d}\widehat{\rho} + E \cdot \ln(\widehat{\rho}). \tag{68}$$

It is easy to check that when $V(\widehat{\rho})$ satisfies

$$V_0 \leqslant V(\widehat{\rho}) \leqslant V_1 \qquad \widehat{\rho} \text{ large} \quad V_0, V_1 \in R \tag{69}$$

then

$$W(\widehat{\rho}) \leqslant V(\widehat{\rho}) + (E - V_0)\ln(\widehat{\rho}). \tag{70}$$

Therefore, choosing $E < V_0$ we get $W(\widehat{\rho}) < A$, $A \in R$ (remember that by equation (69) $V(\widehat{\rho}) \leqslant V_1$), and therefore escape to infinity occurs on any of the invariant planes of equation (63).

All the potentials of section 2 (see table 1) satisfy equation (69) and therefore admit escape when perturbed by constant electric fields.

The reader will check that escape can also be obtained when $\ddot{\mathbf{x}} = -\nabla(V(\rho) + \tilde{V}(\rho)p(\theta))$, (ρ, θ, ϕ) being the standard spherical coordinates in R^3, under suitable assumptions on $\tilde{V}(\rho)$.

Note also that equation (64) for $V(\widehat{\rho}) = \frac{-1}{\widehat{\rho}}$ is integrable [17, 18] since a first integral, in addition to the energy, exists [19].

Nevertheless the KAM theory of integrable systems [18] is useless in the context of our work and *cannot* be applied to equation (64) (for $V(\widehat{\rho}) = \frac{-1}{\widehat{\rho}}$ and ε a small perturbation) since conservation of *unbounded* motions (under perturbations) is outside the reach of KAM theory.

To conclude this section let us mention that we have studied the escape solutions of the equations of type

$$\ddot{\mathbf{x}} = -\nabla V(\rho) + \dot{\mathbf{x}} \wedge \mathbf{B} \qquad \mathbf{x} \in R^3 \quad \mathbf{B} = \text{constant} \tag{71}$$

where $\wedge$ denotes the standard vector product, with little success.

Note that for $\mathbf{B} = (0, 0, B)$ the $z = 0$ plane is invariant under equation (71). On this plane equation (71) becomes

$$(\ddot{x}, \ddot{y}) = -\nabla V\left(\sqrt{x^2 + y^2}\right) + B(\dot{y}, -\dot{x}). \tag{72}$$

This last equation has the two first integrals

$$E = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(\rho) \qquad L = x\dot{y} - y\dot{x} + \frac{B}{2}(x^2 + y^2). \tag{73}$$

Upon introducing polar coordinates and eliminating $\dot{\theta}$ we get

$$\frac{1}{2\rho^2}\left(L - \frac{B\rho^2}{2}\right)^2 + V(\rho) \leqslant E. \tag{74}$$

Equation (74) implies that ρ is bounded when $V(\rho)$ is bounded. Therefore *no* scattering solutions of equation (71) can be obtained by this method.

Another open problem concerning escape solutions appears when an additional electric term $(\varepsilon, 0, 0)$ is introduced in equation (71).

6. Final remarks and open problems

Escape to infinity has been detected (see sections 2, 3 and 5) in some physical systems of type

$$\ddot{\mathbf{x}} = -\nabla V(\mathbf{x}, t) + \varepsilon(\mathbf{x}) + \bar{\bar{A}}(\mathbf{x})\dot{\mathbf{x}} \qquad \mathbf{x} \in R^n \quad (n = 2, 3) \qquad \mathbf{x} = (x_i). \tag{75}$$

A more difficult task is to get conditions on the e.m.f. $(\varepsilon(\mathbf{x}), \mathbf{B}(\mathbf{x}))$ or $(\varepsilon(t), \mathbf{B}(t))$ or even $(\varepsilon(\mathbf{x}, t), \mathbf{B}(\mathbf{x}, t))$ in order that equation (75) has unbounded solutions for $t > 0$.

An interesting open problem is that of showing or disproving that equation

$$\ddot{\mathbf{x}} = -\nabla V(\mathbf{x}) \qquad \mathbf{x} = (x, y) \in R^2 \tag{76}$$

where $V(x, y)$ is bounded at infinity ($V(x, y) \leqslant K$ when ρ is large), admits scattering solutions, as happens for the one-dimensional equation $\ddot{x} = -V'(x)$ or for central potentials $V(x, y) = V(\|(x, y)\|)$. We have tried getting a counterexample of a bounded $V(x, y)$ with every solution of equation (76) $\mathbf{x}(t; \mathbf{x}_0, \dot{\mathbf{x}}_0)$ bounded, for arbitrary $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$, without success.

There remains the open problem of studying relativistic escape to infinity when a central potential (or superposition of different central potentials) and a constant electromagnetic field $(\varepsilon, \mathbf{B})$ are present. The methods developed in this paper seem to fail for this equation and possibly a different technique is necessary to ascertain escape to infinity.

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Appendix

Let us write equations (17) and (18) in the form

$$\ddot{\rho} = a(\rho) \tag{A.1}$$

$$\ddot{\tilde{\rho}} = a(\tilde{\rho}) + p(t) \qquad p(t) \geqslant 0 \qquad p(0) > 0. \tag{A.2}$$

Remember that $a(\rho)$ (see equation (12)) is negative when ρ is large, and $a(\rho)$ is C^w when the potentials V_i are C^w.

Note that we have explicitly assumed in equation (A.2) that for a certain $t = t_0 = 0$ $p(0)$ is positive. But this is obvious since otherwise $p(t) = 0$ identically and equation (A.2) would become equation (A.1).

We now assume that

$$a'(\rho) = -W''(\rho) > 0. \tag{A.3}$$

Note that $W''(\rho)$ is given by (see equation (16))

$$W''(\rho) = \sum_{i=1}^{N} V_i''(\rho - M) \tag{A.4}$$

and equation (A.3) holds on account of conditions (8*a*) (recall that we consider ρ large). Remember that the potentials V_i such that $V_i' < 0$ (for large ρ) have been eliminated in the proof (see section 2).

Let us now prove that when ρ(t) satisfies equation (A.1) and

$$\rho(0) = \rho_0 \qquad \dot{\rho}(0) = \dot{\rho}_0 \qquad \rho(t) = \text{unbounded} \quad \text{for} \quad t > 0 \tag{A.5}$$

then the corresponding solution $\tilde{\rho}(t)$ of (A.2), with initial conditions

$$\tilde{\rho}(0) = \rho_0 \qquad \dot{\tilde{\rho}}(0) = \dot{\rho}_0 \tag{A.6}$$

is also unbounded.

Indeed, subtracting equations (A.2) and (A.1) we get

$$\ddot{\tilde{\rho}}(t) - \ddot{\rho}(t) = a(\tilde{\rho}(t)) - a(\rho(t)) + p(t) \tag{A.7}$$

$\rho(t)$ and $\tilde{\rho}(t)$ being the solutions to (A.1) and (A.2) respectively with the same initial conditions $(\rho_0, \dot{\rho}_0)$.

Fixing $t = t_i$ and applying the mean value theorem to the term $a(\tilde{\rho}(t)) - a(\rho(t))$ of (A.7) we get

$$\ddot{\tilde{\rho}}(t_i) - \ddot{\rho}(t_i) = a'(i(t_i))(\tilde{\rho}(t_i) - \rho(t_i)) + p(t_i) \stackrel{\text{def}}{=} q(t_i)(\tilde{\rho}(t_i) - \rho(t_i)) + p(t_i) \tag{A.8}$$

$i(t_i)$ being a real number between $\rho(t_i)$ and $\tilde{\rho}(t_i)$.

Note that for a generic t we can write $q(t)$ in the form

$$q(t) = \frac{a(\tilde{\rho}(t)) - a(\rho(t))}{\tilde{\rho}(t) - \rho(t)} \tag{A.9}$$

$q(t)$ being a C^0 function of t since

(i) $a(\rho)$ is derivable (it is in fact analytic for large ρ if the potentials $V_i(u)$ are analytic),

(ii) $\frac{a(x)-a(y)}{x-y} = a'(x)$ when $x = y$ $(x, y \in R)$ is continuous.

Note that by (A.3) $q(t)$ is positive.

Setting $\tilde{\rho}(t) - \rho(t) = x(t)$, we can write (A.8) in the form (see (A.2))

$$\left.\begin{array}{ll} \ddot{x}(t) = q(t)x(t) + p(t) & \\ x(0) = 0 & \dot{x}(0) = 0 \\ q(t) \in C^0 & p(t) \in C^w \\ q(t) > 0 & p(0) > 0 \end{array}\right\} \tag{A.10}$$

$p(t)$ being analytic when the potentials $V_i(u)$ are analytic.

Let us now prove that any of the C^2 solutions of (A.10) [20] satisfies (note that as $q(t)$ is only a continuous function then the Peano theorem [20] only guarantees the existence of solutions but not their unicity)

$$x(t) \geqslant 0 \qquad t > 0. \tag{A.11}$$

Indeed from (A.10) we get $\ddot{x}(0) = p(0) > 0$ and therefore $\ddot{x}(t) > 0$, $\dot{x}(t) > 0$ and $x(t) > 0$ for $t \in (0, t_1]$, $t_1 > 0$.

Assume that for some $t_2 > t_1 > 0$ we have $x(t_2) < 0$; then there exists $\bar{t} \in (0, t_2)$ such that $\dot{x}(\bar{t}) = 0$ and $x(t) > 0$ for $0 < t < \bar{t}$. Therefore by the mean value theorem we get

$$\dot{x}(\bar{t}) - \dot{x}(0) = \ddot{x}(\bar{\bar{t}})\bar{t} \qquad \bar{\bar{t}} \in (0, \bar{t}) \tag{A.12}$$

and therefore $\ddot{x}(\bar{\bar{t}}) = 0$. But this contradicts equation (A.10) and therefore $x(t) > 0 \;\; \forall t > 0$.

As $\tilde{\rho}(t) = \rho(t) + x(t)$ and $x(t) > 0 \; \forall t > 0$ we obtain that $\rho(t)$ unbounded implies $\tilde{\rho}(t)$ also unbounded, as we desired to prove.

References

[1] Fulcher P, Davis F and Rowe A 1976 *Am. J. Phys.* **44** 956
Menun V and Agraval D 1986 *Am. J. Phys.* **54** 752
Ozorio A, Leon N, Mehta M and Marston C 1990 *Physica* D **46** 265
Jaffe C, Farrelly D and Uzer T 1999 *Phys. Rev.* A **60** 3833
Jaffe C, Farrelly D and Uzer T 2000 *Phys. Rev. Lett.* **84** 610
Wiggins S, Wiesenfeld L, Jaffe C and Uzer T 2001 *Phys. Rev. Lett.* **86** 5478
[2] Goriely A and Hyde C 1998 *Phys. Lett.* A **250** 31
Matsumo Y 1992 *J. Math. Phys.* **33** 412
[3] Goldstein H, Poole C and Safko J 2002 *Classical Mechanics* (San Francisco, CA: Addison-Wesley)
[4] Gascon F G and Peralta-Salas D 2000 *J. Phys. A: Math. Gen.* **33** 5361
[5] Mie G 1903 *Ann. Phys., Lpz* **11** 657
Ashcroft N W and Mermin N D 1976 *Solid State Physics* (Orlando, FL: Saunders)
[6] Morse P M 1929 *Phys. Rev.* **34** 57
Wadi H and Wiesenfeld L 1997 *Phys. Rev.* E **55** 271
[7] Peskin M E and Schroeder D V 1995 *An Introduction to Quantum Field Theory* (Reading, MA: Addison-Wesley)
[8] Williams W S C 1991 *Nuclear and Particle Physics* (New York: Oxford University Press)
[9] Eckart C 1930 *Phys. Rev.* **35** 1303
[10] Hylleraas E A 1937 *Z. Phys.* **107** 258
[11] Menun V and Agraval D 1986 *Am. J. Phys.* **54** 752
Fulcher L, Davis B and Rowe D 1976 *Am. J. Phys.* **44** 956
Gutkin E 1985 *Physica* D **16** 398
Gutkin E 1985 *Physica* D **17** 235
Gutkin E 1987 *J. Math. Phys.* **28** 351
Marchioro C 1970 *J. Math. Phys.* **11** 2193
Moauro V and Negrini P 1991 *J. Differ. Eqns.* **90** 61
Fusco G and Oliva W 1997 *J. Differ. Eqns.* **135** 16
Halperin G 1982 *Commun. Math. Phys.* **84** 547
Hubacher A 1989 *Commun. Math. Phys.* **123** 353
Vaserstein L 1979 *Commun. Math. Phys.* **69** 31

[12] Cartan H 1941 *Bull. Soc. Math. France* **69** 71
Cartan H 1945 *Bull. Soc. Math. France* **73** 74
[13] Apostol T 1982 *Calculus* (Barcelona: Reverté)
Bartle R and Sherbert D 1982 *Introduction to Real Analysis* (New York: Wiley)
[14] Panofsky W and Phillips M 1969 *Classical Electricity and Magnetism* (Reading, MA: Addison-Wesley)
Arfken G 1985 *Mathematical Methods for Physicists* (Orlando, FL: Academic)
[15] Schonfeld J 1980 *J. Math. Phys.* **21** 2528
Leach P and Gorringe V 1990 *J. Phys. A: Math. Gen.* **23** 2765
[16] Arfken G 1985 *Mathematical Methods for Physicists* (Orlando, FL: Academic)
Lange O and Raab R 1991 *Operator Methods in Quantum Mechanics* (Oxford: Clarendon)
Campuzano O, Nuñez H, Salas A and Sánchez G 1995 *Eur. J. Phys.* **16** 220
Lopez J, Morales J and Ovando G 1998 *Eur. J. Phys.* **19** 401
[17] Lopez J, Morales J and Ovando G 1998 *Eur. J. Phys.* **19** 401
[18] Arnold V I 1963 *Russ. Math. Surveys* **18** 9
Marshall I and Wojciechowki S 1988 *J. Math. Phys.* **29** 1338
[19] Pars L 1965 *A Treatise on Analytical Dynamics* (London: Heisemann)
Landau L and Lifchitz E 1972 *Mecanique* (Moscow: MIR)
[20] Birkhoff G and Rota G C 1978 *Ordinary Differential Equations* 3rd edn (New York: Wiley)